Cyclic cohomology and Hopf algebras

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Dedicated to the memory of Moshé Flato

Abstract

We show by a direct computation that, for any Hopf algebra with a modulus-like character, the formulas first introduced in [CM] in the context of characteristic classes for actions of Hopf algebras, do define a cyclic module. This provides a natural generalization of Lie algebra cohomology to the general framework of Noncommutative Geometry, which covers the case of the Hopf algebra associated to n-dimensional transverse geometry [CM] as well as the function algebras of the classical quantum groups.

Introduction

We shall concentrate in this paper on the interplay between two basic concepts of Noncommutative geometry. The first is cyclic cohomology which plays the same role in Noncommutative geometry as De Rham cohomology plays in differential geometry. The second is Hopf algebras whose actions on noncommutative algebras are analoguous to Lie group actions on ordinary manifolds.

We shall show by a direct and elementary computation that, for any Hopf algebra with a modulus-like character, the natural cosimplicial module associated to the subjacent coalgebra structure can be upgraded to a *cyclic module* (or, rather, a module over the *cyclic category* Λ, cf. [C, III.A]), by invoking both the product and the antipode. This cyclic module was first introduced in [CM] in the context of characteristic classes for actions of Hopf algebras, under a certain condition of existence of sufficiently nondegenerate actions, which made the verification of the axioms tautological. The fact that the latter assumption was superfluous has also been remarked by M. Crainic [Cr], who recasted our construction in the framework of the Cuntz-Quillen

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formalism [CQ].

I Cyclic cohomology and the cyclic category

The role of cyclic cohomology in Noncommutative geometry can be understood at several levels. In its simplest guise it is a construction of invariants of K-theory, extending to the general framework the Chern-Weil characteristic classes of vector bundles and allowing for concrete computations on Noncommutative spaces. For starters one should prove for oneself the following simple algebraic statement which extends to higher dimension the obvious properties of the K-theory invariant provided by a trace τ on a noncommutative algebra A, by means of the equality,

$$< E\,, \tau >= \tau(E) \tag{1}$$

for any idempotent E, $E^2 = E$, $E \in M_q(A)$, where the trace τ is extended to the algebra $M_q(A)$ of matrices over A by,

$$\tau((a_{i,j})) \; = \sum \tau(a_{i,i}) \tag{2}$$

To pass from this 0-dimensional situation to, say, dimension 2, one considers a trilinear form τ on the algebra A which possesses the following compatibility with the algebra structure, reminiscent of the properties of a trace,

(3)

$$\tau(a^1, a^2, a^0) \; = \tau(a^0, a^1, a^2)$$
$$\tau(a^0a^1, a^2, a^3) \; - \tau(a^0, a^1a^2, a^3) \; + \tau(a^0, a^1, a^2a^3) \; - \tau(a^3a^0, a^1, a^2) = 0. .$$
$$\forall a^j \in A$$

The statement then asserts that for each idempotent E, $E^2 = E$, $E \in M_q(A)$, the scalar $\tau(E,E,E)$ remains constant when E is deformed among the idempotents of $M_q(A)$. This homotopy invariance of the resulting pairing between cyclic cocycles of arbitrary dimension (i.e. multilinear forms on A fulfilling the n-dimensional analogue of (3)) and K-theory is the starting point of cyclic cohomology.

At the conceptual level, cyclic cohomology is obtained as an Ext functor by linearisation of the non-additive category of algebras and algebra homomorphisms ([C2]) using the additive category of Λ-modules where Λ is the cyclic category.

The cyclic category is a small category which can be defined by generators and relations. It has the same objects as the small category Δ of totally ordered finite sets and increasing maps. Let us recall the presentation of Δ. It has one object $[n]$ for each integer n, and is generated by faces δ_i, $[n-1] \to [n]$

(the injection that misses i), and degeneracies $\sigma_j, [n+1] \to [n]$ (the surjection which identifies j with $j+1$), with the relations,

(4) $$\delta_j\, \delta_i = \delta_i\, \delta_{j-1} \text{ for } i < j\,,\ \ \sigma_j\, \sigma_i = \sigma_i\, \sigma_{j+1} \qquad i \leq j$$

$$\sigma_j\, \delta_i = \begin{cases} \delta_i\, \sigma_{j-1} & i < j \\ 1_n & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1}\, \sigma_j & i > j+1\,. \end{cases}$$

To obtain Λ one adds for each n a new morphism $\tau_n, [n] \to [n]$ such that,

(5) $$\begin{aligned} &\tau_n\, \delta_i = \delta_{i-1}\, \tau_{n-1} \quad 1 \leq i \leq n, \quad \tau_n\, \delta_0 = \delta_n \\ &\tau_n\, \sigma_i = \sigma_{i-1}\, \tau_{n+1} \quad 1 \leq i \leq n, \quad \tau_n\, \sigma_0 = \sigma_n\, \tau_{n+1}^2 \\ &\tau_n^{n+1} = 1_n\,. \end{aligned}$$

The small category Λ is in fact best obtained as a quotient of the following category $E\,\Lambda$. The latter has one object $(\mathbb{Z}, n)$ for each n and the morphisms $f : (\mathbb{Z}, n) \to (\mathbb{Z}, m)$ are non decreasing maps, $(n, m \geq 1)$

(6) $$f : \mathbb{Z} \to \mathbb{Z}\,,\ \ f(x+n) = f(x) + m \qquad \forall\, x \in \mathbb{Z}\,.$$

One has $\Lambda = (E\,\Lambda)/\mathbb{Z}$ for the obvious action of $\mathbb{Z}$ by translation. The original definition of Λ (cf. [C2]) used homotopy classes of non decreasing maps from S^1 to S^1 of degree 1, mapping $\mathbb{Z}/n$ to $\mathbb{Z}/m$ and is trivially equivalent to the above.

Given an algebra A one obtains a module over the small category Λ by assigning to each integer $n \geq 0$ the vector space C^n of $n+1$-linear forms $\varphi(x^0, \ldots, x^n)$ on A, while the basic operations are given by

(7) $$\begin{aligned} (\delta_i\, \varphi)(x^0, \ldots, x^n) &= \varphi(x^0, \ldots, x^i\, x^{i+1}, \ldots, x^n) \qquad i = 0, 1, \ldots, n-1 \\ (\delta_n\, \varphi)(x^0, \ldots, x^n) &= \varphi(x^n\, x^0, x^1, \ldots, x^{n-1}) \\ (\sigma_j\, \varphi)(x^0, \ldots, x^n) &= \varphi(x^0, \ldots, x^j, 1, x^{j+1}, \ldots, x^n) \quad j = 0, 1, \ldots, n \\ (\tau_n\, \varphi)(x^0, \ldots, x^n) &= \varphi(x^n, x^0, \ldots, x^{n-1})\,. \end{aligned}$$

In the first two lines $\delta_i : C^{n-1} \to C^n$. In the third line $\sigma_i : C^{n+1} \to C^n$. Note that $(\sigma_n\, \varphi)(x^0, \ldots, x^n) = \varphi(x^0, \ldots, x^n, 1)$, $(\sigma_0\, \varphi)(x^0, \ldots, x^n) = \varphi(x^0, 1, x^1, \ldots,\, x^n)$.

These operations satisfy the relations (4) and (5). This shows that any algebra A gives rise canonically to a Λ-module and allows ([C2][L]) to interpret the cyclic cohomology groups $HC^n(A)$ as Ext^n functors. All of the general properties of cyclic cohomology such as the long exact sequence relating it to Hochschild cohomology are shared by Ext of general Λ- modules and can be

attributed to the equality of the classifying space $B\Lambda$ of the small category Λ with the classifying space BS^1 of the compact one-dimensional Lie group S^1.

II Characteristic classes for actions of Hopf algebras

Hopf algebras arise very naturally from their actions on noncommutative algebras. Given an algebra A, an action of the Hopf algebra $\mathcal{H}$ on A is given by a linear map,

$$\mathcal{H} \otimes A \to A \ , \ \ h \otimes a \to h(a)$$

satisfying $h_1(h_2\, a) = (h_1\, h_2)(a) \quad \forall\, h_i \in \mathcal{H}$, $a \in A$ and

$$h(ab) = \sum h_{(1)}\,(a)\, h_{(2)}\,(b) \qquad \forall\, a, b \in A\,,\ \ h \in \mathcal{H}\,. \tag{1}$$

where the coproduct of h is,

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)} \tag{2}$$

In concrete examples, the algebra A appears first, together with linear maps $A \to A$ satisfying a relation of the form (1) which dictates the Hopf algebra structure. We refer to [CM] for an application of this construction to the leaf space of foliations.

The theory of characteristic classes for actions of $\mathcal{H}$ extends the construction ([C3]) of cyclic cocycles from a Lie algebra of derivations of a C^* algebra A, together with an *invariant trace* τ on A.

In order to cover the nonunimodular case which does appear in the simplest examples, we fix a character δ of $\mathcal{H}$ which will play the role of the module of locally compact groups.

We then introduce the twisted antipode,

$$\widetilde{S}(y) = \sum \delta(y_{(1)})\, S(y_{(2)})\ ,\ \ y \in \mathcal{H}\,,\ \ \Delta\, y = \sum y_{(1)} \otimes y_{(2)}\,. \tag{3}$$

One has $\widetilde{S}(y) = S(\sigma(y))$ where σ is the automorphism $\sigma = (\delta \otimes 1) \circ \Delta : \mathcal{H} \to \mathcal{H}$.

Definition 1. *We shall say that a trace τ on A is δ-invariant under the action of $\mathcal{H}$ iff the following holds,*

$$\tau(h(a)b) = \tau(a\, \widetilde{S}(h)(b)) \qquad \forall\, a, b \in A\,,\ \ h \in \mathcal{H}\,.$$

We have shown in ([CM]) that the definition of the cyclic complex $HC^*_\delta(\mathcal{H})$ is uniquely dictated in such a way that the following holds,

Proposition 2. ([CM]) *Let τ be a δ-invariant trace on A, then the following defines a canonical map from $HC^*_\delta(\mathcal{H})$ to $HC^*(A)$,*

$$\gamma(h^1 \otimes \ldots \otimes h^n) \in C^n(A)\,,\ \ \gamma(h^1 \otimes \ldots \otimes h^n)(x^0, \ldots, x^n) =$$

$$\tau(x^0\, h^1(x^1) \ldots h^n(x^n))\,.$$

In ([CM]) we needed to assume the existence of enough such actions of $\mathcal{H}$ in order to check that the formulas were actually defining a cyclic module. We shall show below by a direct and elementary computation that, for any Hopf algebra with a modulus-like character δ as above, the construction of [CM] does yield a cyclic module.

III The cyclic module of a Hopf algebra

In this section we shall associate a cyclic complex (in fact a Λ-module, where Λ is the cyclic category), to any Hopf algebra together with a character δ such that the δ-twisted antipode is an involution. The resulting cyclic cohomology appears to be the natural candidate for the analogue of Lie algebra cohomology in the context of Hopf algebras, where both the Hochschild cohomology (also called Sweedler cohomology) or the transposed (also called Harrison cohomology) give too naive results.

Let $\mathcal{H}$ be a Hopf algebra (over $\mathbb{C}$) with unit map $\eta : \mathbb{C} \to \mathcal{H}$, counit $\varepsilon : \mathcal{H} \to \mathbb{C}$ and antipode $S : \mathcal{H} \to \mathcal{H}$,

$$S * I = I * S = \eta\,\varepsilon.$$

We fix a character $\delta : \mathcal{H} \to \mathbb{C}$, which will play the role of the modular function of a locally compact group. With the usual coproduct notation

$$\Delta h = \sum_{(h)} h_{(1)} \otimes h_{(2)} \quad , \quad h \in \mathcal{H}\,,$$

we introduce the δ-twisted antipode

$$\widetilde{S}(h) = \sum_{(h)} \delta(h_{(1)})\; S(h_{(2)}) \quad , \quad h \in \mathcal{H}\,. \tag{1}$$

The elementary properties of S imply immediately that $\widetilde{S}$ is an algebra antihomomorphism

$$\begin{aligned} &\widetilde{S}(h^1\, h^2) = \widetilde{S}(h^2)\, \widetilde{S}(h^1) \quad , \quad \forall\, h^1, h^2 \in \mathcal{H} \\ &\widetilde{S}(1) = 1\,, \end{aligned} \tag{2}$$

a coalgebra twisted antimorphism

$$\Delta\, \widetilde{S}(h) = \sum_{(h)} S(h_{(2)}) \otimes \widetilde{S}(h_{(1)}) \quad , \quad \forall\, h \in \mathcal{H}\,; \tag{3}$$

and also that it satisfies

$$\varepsilon \circ \widetilde{S} = \delta\,. \tag{4}$$

By transposing the standard simplicial operators underlying the Hochschild homology complex of an algebra, one associates to $\mathcal{H}$, viewed only as a coalgebra, the following natural cosimplicial module: $\{\mathcal{H}^{\otimes n}\}_{n\geq 1}$, with face operators $\delta_i : \mathcal{H}^{\otimes n-1} \to \mathcal{H}^{\otimes n}$,

$$
\begin{aligned}
&\delta_0(h^1 \otimes \ldots \otimes h^{n-1}) = 1 \otimes h^1 \otimes \ldots \otimes h^{n-1} \,, \\
(5) \qquad &\delta_j(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^n \,, \ \forall\, 1 \leq j \leq n-1 \,, \\
&\delta_n(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes h^{n-1} \otimes 1
\end{aligned}
$$

and degeneracy operators $\sigma_i : \mathcal{H}^{\otimes n+1} \to \mathcal{H}^{\otimes n}$,

$$
(6) \qquad \sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) = h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1} \ , \ 0 \leq i \leq n \,.
$$

In [CM, § 7] the remaining two essential features of a Hopf algebra – *product* and *antipode* – are brought into play, to define the *cyclic operators* $\tau_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$,

$$
(7) \qquad \tau_n(h^1 \otimes \ldots \otimes h^n) = (\Delta^{n-1}\, \widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^n \otimes 1 \,.
$$

Theorem 3. *Let $\mathcal{H}$ be a Hopf algebra endowed with a character $\delta \in \mathcal{H}^*$ such that the corresponding twisted antipode* (1) *is an involution:*

$$
(8) \qquad \widetilde{S}^2 = I \,.
$$

Then $\mathcal{H}_\delta^\natural = \{\mathcal{H}^{\otimes n}\}_{n\geq 1}$ equipped with the operators given by (5)–(7) *defines a module over the cyclic category Λ.*

Proof. One has to check the relations

$$
(9) \qquad
\begin{aligned}
&\tau_n\, \delta_i = \delta_{i-1}\, \tau_{n-1} \ , \ 1 \leq i \leq n \,, \\
&\tau_n\, \delta_0 = \delta_n \,,
\end{aligned}
$$

$$
(10) \qquad
\begin{aligned}
&\tau_n\, \sigma_i = \sigma_{i-1}\, \tau_{n+1} \ , \ 1 \leq i \leq n \,, \\
&\tau_n\, \sigma_0 = \sigma_n\, \tau_{n+1}^2 \,,
\end{aligned}
$$

$$
(11) \qquad \tau_n^{n+1} = I_n \,.
$$

It is the latter which poses a technical challenge. To size it up, let us first look at the case $n = 2$.

In what follows we shall only use the basic properties of the product, the coproduct, the antipode and of the twisted antipode (cf. (1)–(4)). We shall

also employ the usual notational conventions for the Hopf algebra calculus (cf. [S]).

To begin with,

$$\begin{aligned}\tau_2(h^1\otimes h^2) &= \Delta\,\widetilde{S}(h^1)\cdot h^2\otimes 1 = \\ &= \sum \widetilde{S}(h^1)_{(1)}\,h^2\otimes \widetilde{S}(h^1)_{(2)} \\ &= \sum S(h^1_{(2)})\,h^2\otimes \widetilde{S}(h^1_{(1)})\,.\end{aligned}$$

Its square is therefore:

$$\begin{aligned}\tau_2^2(h^1\otimes h^2) &= \sum S(S(h^1_{(2)})_{(2)}\,h^2_{(2)})\,\widetilde{S}(h^1_{(1)})\otimes \widetilde{S}(S(h^1_{(2)})_{(1)}\,h^2_{(1)}) \\ &= \sum S(S(h^1_{(2)(1)})\,h^2_{(2)})\,\widetilde{S}(h^1_{(1)})\otimes \widetilde{S}(S(h^1_{(2)(2)})\,h^2_{(1)}) \\ &= \sum S(h^2_{(2)})\,(S\circ S)\,(h^1_{(2)(1)})\,\widetilde{S}(h^1_{(1)})\otimes \widetilde{S}(h^2_{(1)})\,(\widetilde{S}\circ S)\,(h^1_{(2)(2)}) \\ &= \sum S(h^2_{(2)})\,\boxed{S(S(h^1_{(1)(2)}))\,\widetilde{S}(h^1_{(1)(1)})}\,\otimes \widetilde{S}(h^2_{(1)})\,\widetilde{S}(S(h^1_{(2)}))\,.\end{aligned}$$

The term in the box is computed as follows. With $k = h^1_{(1)}$, one has

$$\begin{aligned}\sum S(S(k_{(2)}))\,\widetilde{S}(k_{(1)}) &= \sum S(S(k_{(2)}))\,\delta(k_{(1)(1)})\,S(k_{(1)(2)}) \\ &= \sum S(S(k_{(2)(2)})\,\delta(k_{(1)})\,S(k_{(2)(1)}) = \\ &= \sum \delta(k_{(1)})\,S\left(\sum k_{(2)(1)}\,S(k_{(2)(2)})\right) \\ &= \sum \delta(k_{(1)})\,S(\varepsilon(k_{(2)})\,1) = \\ &= \sum \delta(k_{(1)})\,\varepsilon(k_{(2)}) = \delta\left(\sum k_{(1)}\,\varepsilon(k_{(2)})\right) \\ &= \delta(k)\,.\end{aligned}$$

It follows that

$$\begin{aligned}\tau_2^2(h^1\otimes h^2) &= \sum S(h^2_{(2)})\,\underbrace{\delta(h^1_{(1)})\otimes \widetilde{S}(h^2_{(1)})\,\widetilde{S}(S(h^1_{(2)}))} \\ &= \sum S(h^2_{(2)})\otimes \widetilde{S}(h^2_{(1)})\,\widetilde{S}(\widetilde{S}(h^1)) = \\ &= \sum S(h^2_{(2)})\otimes \widetilde{S}(h^2_{(1)})\,h^1\,,\end{aligned}$$

where we have used first (1) then (8). Thus

$$\begin{aligned}\tau_2^2(h^1\otimes h^2) &= \sum S(h^2_{(2)})\otimes \widetilde{S}(h^2_{(1)})\cdot 1\otimes h^1 \\ &= \Delta\sum \widetilde{S}(h^2)\cdot 1\otimes h^1\,.\end{aligned}$$

In a similar fashion,

$$\begin{aligned}\tau_2^3(h^1\otimes h^2) &= \sum S(S(h^2_{(2)})_{(2)})\,\widetilde{S}(h^2_{(1)})\,h^1\otimes \widetilde{S}(S(h^2_{(2)})_{(1)}) \\ &= \sum S(S(h^2_{(2)(1)}))\,\widetilde{S}(h^2_{(1)})\,h^1\otimes \widetilde{S}(S(h^2_{(2)(2)})) \\ &= \sum S(S(h^2_{(1)(2)}))\,\widetilde{S}(h^2_{(1)(1)})\,h^1\otimes \widetilde{S}(S(h^2_{(2)})) \\ &= \sum \delta(h^2_{(1)})\,h^1\otimes \widetilde{S}(S(h^2_{(2)})) = \\ &= \sum h^1\otimes \widetilde{S}^2(h^2) = h^1\otimes h^2\,.\end{aligned}$$

We now pass to the general case. With the standard conventions of notation,

$$\tau_n(h^1 \otimes h^2 \otimes \ldots \otimes h^n) = \Delta^{(n-1)}\, \widetilde{S}(h^1) \cdot h^2 \otimes \ldots \otimes h^n \otimes 1$$
$$= \sum S(h^1_{(n)})\, h^2 \otimes S(h^1_{(n-1)})\, h^3 \otimes \ldots \otimes S(h^1_{(2)})\, h^n \otimes \widetilde{S}(h^1_{(1)})\,.$$

Upon iterating once

$$\begin{aligned}
\tau_n^2(h^1 \otimes \ldots \otimes h^n) = &\sum S(S(h^1_{(n)})_{(n)}\, h^2_{(n)}))\, S(h^1_{(n-1)})\, h^3 \otimes \\
&\otimes S(S(h^1_{(n)})_{(n-1)}\, h^2_{(n-1)}))\, S(h^1_{(n-2)})\, h^4 \otimes \ldots \\
\ldots &\otimes S(S(h^1_{(n)})_{(2)}\, h^2_{(2)}))\, \widetilde{S}(h^1_{(1)}) \otimes \widetilde{S}(S(h^1_{(n)})_{(1)}\, h^2_{(1)}) \\
= &\sum S(h^2_{(n)})\, S(S(h^1_{(n)(1)}))\, S(h^1_{(n-1)})\, h^3 \otimes \\
&\otimes S(h^2_{(n-1)})\, S(S(h^1_{(n)(2)}))\, S(h^1_{(n-2)})\, h^4 \otimes \ldots \\
\ldots &\otimes S(h^2_{(2)})\, S(S(h^1_{(n)(n-1)}))\, \widetilde{S}(h^1_{(1)}) \otimes \\
&\otimes \widetilde{S}(h^2_{(1)})\, \widetilde{S}(S(h^1_{(n)(n)})) = \\
= &\sum S(h^2_{(n)})\, S(h^1_{(n-1)}\, S(h^1_{(n)}))\, h^3 \otimes \\
&\otimes S(h^2_{(n-1)})\, S(h^1_{(n-2)}\, S(h^1_{(n+1)}))\, h^4 \otimes \ldots \\
\ldots &\otimes S(h^2_{(2)})\, S(S(h^1_{(2n-2)})) \cdot \widetilde{S}(h^1_{(1)}) \otimes \\
&\otimes \widetilde{S}(h^2_{(1)})\, \widetilde{S}(S(h^1_{(2n-1)}))\,.
\end{aligned}$$

We pause to note that

$$\sum h^1_{(n-1)}\, S(h^1_{(n)}) = \sum h^1_{(n-1)(1)}\, S(h^1_{(n-1)(2)})$$

equals

$$\varepsilon\,(h^1_{(n-1)})\, 1\,,$$

after resetting the indexation. Next

$$\sum \varepsilon(h^1_{(n-1)})\, h^1_{(n-2)}$$

gives $h^1_{(n-2)}$ after another resetting. In turn

$$\sum h^1_{(n-2)}\, S(h^1_{(n-1)})$$

equals

$$\varepsilon(h^1_{(n-2)})\, 1\,,$$

and the process continues.

In the last step,

$$\begin{aligned}
&\sum S(h^2_{(n)})\, h^3 \otimes S(h^2_{(n-1)})\, h^4 \otimes \ldots \\
\ldots \otimes\, &S(h^2_{(2)}) \boxed{\, S(S(h^1_{(2)}))\, \delta(h^1_{(1)(1)})\, S(h^1_{(1)(2)}) \,} \otimes \\
&\otimes \widetilde{S}(h^2_{(1)})\, \widetilde{S}(S(h^1_{(3)})) \\
&= \sum S(h^2_{(n)})\, h^3 \otimes S(h^2_{(n-1)})\, h^4 \otimes \ldots \otimes S(h^2_{(2)}) \otimes \widetilde{S}(h^2_{(1)})\, h^1 \\
&= S(h^2_{(n)}) \otimes S(h^2_{(n-1)}) \otimes \ldots \otimes \widetilde{S}(h^2_{(1)}) \cdot h^3 \otimes h^4 \otimes \ldots \otimes 1 \otimes h^1 \\
&= \Delta^{(n-1)}\, \widetilde{S}(h^2) \cdot h^3 \otimes h^4 \otimes \ldots \otimes 1 \otimes h^1\,,
\end{aligned}$$

with the boxed term simplified as before.

By induction, one obtains for any $j = 1, \ldots, n+1$,

$$\tau_n^j(h^1 \otimes \ldots \otimes h^n) = \Delta^{n-1} \widetilde{S}(h^j) \cdot h^{j+1} \otimes \ldots \otimes h^n \otimes 1 \otimes \ldots \otimes h^{j-1} \,,$$

in particular

$$\tau_n^{n+1}(h^1 \otimes \ldots \otimes h^n) = \Delta^{n-1} \widetilde{S}(1) \cdot h^1 \otimes \ldots \otimes h^n = h^1 \otimes \ldots \otimes h^n \,.$$

The verification of the compatibility relations (9), (10) is straightforward. Indeed, starting with the compatibility with the face operators, one has:

$$\begin{aligned}
\tau_n \, \delta_0(1 \otimes h^1 \otimes \ldots \otimes h^{n-1}) &= \tau_n(1 \otimes h^1 \otimes \ldots \otimes h^{n-1}) = \\
&= \Delta^{n-1} \widetilde{S}(1) \cdot h^1 \otimes \ldots \otimes h^{n-1} \otimes 1 \\
&= h^1 \otimes \ldots \otimes h^{n-1} \otimes 1 \\
&= \delta_n(h^1 \otimes \ldots \otimes h^{n-1}) \,,
\end{aligned}$$

then

$$\begin{aligned}
\tau_n \, \delta_1(h^1 \otimes \ldots \otimes h^{n-1}) &= \tau_n \, (\Delta \, h^1 \otimes h^2 \otimes \ldots \otimes h^{n-1}) \\
&= \sum \tau_n(h^1_{(1)} \otimes h^1_{(2)} \otimes h^2 \otimes \ldots \otimes h^{n-1}) \\
&= \sum \Delta^{n-1} \widetilde{S}(h^1_{(1)}) \cdot h^1_{(2)} \otimes h^2 \otimes \ldots \otimes h^{n-1} \otimes 1 = \\
&= \sum S(h^1_{(1)(n)}) \, h^1_{(2)} \otimes S(h^1_{(1)(n-1)}) \, h^2 \otimes \ldots \\
&\quad \otimes S(h^1_{(1)(2)}) \, h^{n-1} \otimes \widetilde{S}(h^1_{(1)(1)}) \\
&= \sum \varepsilon(h^1_{(n)}) \, 1 \otimes S(h^1_{(n-1)}) \, h^2 \otimes \ldots \\
&\quad \otimes S(h^1_{(1)}) \, h^{n-1} \otimes \widetilde{S}(h^1_{(1)}) \\
&= 1 \otimes S(h^1_{(n-1)}) \, h^2 \otimes \ldots \otimes S(h^1_{(1)}) \, h^{n-1} \otimes \widetilde{S}(h^1_{(1)}) \\
&= \delta_0 \, \tau_{n-1} \, (h^1 \otimes \ldots \otimes h^{n-1}) \,,
\end{aligned}$$

and so forth.

Passing now to degeneracies,

$$\begin{aligned}
&\tau_n \, \sigma_0(h^1 \otimes \ldots \otimes h^{n+1}) = \varepsilon(h^1) \, \tau_n(h^2 \otimes \ldots \otimes h^{n+1}) = \\
&= \varepsilon(h^1) \, S(h^2_{(n)}) \, h^3 \otimes \ldots \otimes S(h^2_{(2)}) \, h^{n+1} \otimes \widetilde{S}(h^2_{(1)}) \,,
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\sigma_n \, \tau_{n+1}^2(h^1 \otimes \ldots \otimes h^{n+1}) = \\
&= \sigma_n \left(\sum S(h^2_{(n+1)}) \, h^3 \otimes \ldots \otimes S(h^2_{(2)}) \otimes \widetilde{S}(h^2_{(1)}) \, h^1 \right) \\
&= \sum \varepsilon(\widetilde{S}(h^2_{(1)}) \, h^1) \, S(h^2_{(n+1)}) \, h^3 \otimes \ldots \otimes S(h^2_{(2)}) \\
&= \varepsilon(h^1) \sum \delta(h^2_{(1)}) \, S(h^2_{(n+1)}) \, h^3 \otimes \ldots \otimes S(h^2_{(2)}) \\
&= \varepsilon(h^1) \, S(h^2_{(n)}) \, h^3 \otimes \ldots \otimes S(h^2_{(2)}) \, h^{n+1} \otimes \widetilde{S}(h^2_{(1)}) \,.
\end{aligned}$$

In the next step

$$\begin{aligned}
&\tau_n\,\sigma_1(h^1\otimes\ldots\otimes h^{n+1})=\varepsilon(h^2)\,\tau_n(h^1\otimes h^3\otimes\ldots\otimes h^{n+1})\\
&=\varepsilon(h^2)\cdot\Delta^{n-1}\,\widetilde{S}(h^1)\cdot h^3\otimes\ldots\otimes h^{n+1}\otimes 1\,,
\end{aligned}$$

while on the other hand

$$\begin{aligned}
&\sigma_0\,\tau_{n+1}(h^1\otimes\ldots\otimes h^{n+1})=\\
&\sigma_0(S(h^1_{(n+1)})\,h^2\otimes\ldots\otimes S(h^1_{(2)})\,h^{n+1}\otimes\widetilde{S}(h^1_{(1)}))\\
&=\varepsilon(h^2)\cdot\varepsilon(h^1_{(n+1)})\cdot S(h^1_{(n)})\,h^3\otimes\ldots\otimes S(h^1_{(2)})\,h^{n+1}\otimes\widetilde{S}(h^1_{(1)})\\
&=\varepsilon(h^2)\cdot S(h^1_{(n-1)})\,h^3\otimes\ldots\otimes S(h^1_{(2)})\,h^{n+1}\otimes\widetilde{S}(h^1_{(1)})\,,
\end{aligned}$$

and similarly for $i=2,\ldots n$. ■

The cohomology of the (b,B)-bicomplex corresponding to the cyclic module $\mathcal{H}^\natural_\delta$ is, by definition, the *cyclic cohomology* $H\,C^*_\delta(\mathcal{H})$ of $\mathcal{H}$ relative to the modular character δ.

When $\mathcal{H}=\mathcal{U}(\mathbf{G})$ is the envelopping algebra of a Lie algebra, there is a natural interpretation of the Lie algebra cohomology,

$$H^*(\mathbf{G},\mathbb{C})=H^*(\mathcal{U}(\mathbf{G}),\mathbb{C})$$

where the right hand side is the Hochschild cohomology with coefficients in the $\mathcal{U}(\mathbf{G})$-bimodule $\mathbb{C}$ obtained using the augmentation. In general, given a Hopf algebra $\mathcal{H}$ one can dualise (this is the construction of the Harrison complex), the construction of the Hochschild complex $C^n(\mathcal{H}^*,\mathbb{C})$ where $\mathbb{C}$ is viewed as a bimodule on $\mathcal{H}^*$ using the augmentation, i.e. the counit of $\mathcal{H}^*$. This gives the above operations: $\mathcal{H}^{\otimes(n-1)}\to\mathcal{H}^{\otimes n}$, defining a cosimplicial space.

When applied to the Hopf algebra $\mathcal{H}$ of functions on an affine algebraic group, this dual-Hochschild or Harrison cohomology gives simply the vector space of invariant twisted forms and ignores the group cohomology. The second assertion of the following proposition shows that the cyclic cohomology $H\,C^*_\delta(\mathcal{H})$, gives a highly nontrivial answer, thanks precisely to the action of the B operator ([CM]).

Proposition 4. ([CM]) 1) *The periodic cyclic cohomology $H\,C^*_\delta(\mathcal{H})$, for $\mathcal{H}=\mathcal{U}(\mathbf{G})$ the envelopping algebra of a Lie algebra $\mathbf{G}$ is isomorphic to the Lie algebra homology $H_*(\mathbf{G},\mathbb{C})$ where $\mathbb{C}$ is a $\mathbf{G}$-module using the module δ of G.*
2) *Let $\mathcal{H}=\mathcal{U}(\mathbf{G})_*$ be the Hopf algebra of polynomials in the coordinates on an affine simply connected nilpotent group G. The periodic cyclic cohomology $H\,C^*_\delta(\mathcal{H})$, is isomorphic to the Lie algebra cohomology of $\mathbf{G}$ with trivial coefficients.*

We refer to [CM] for the proof which was also reproduced in [Cr].

Finally we should point out that the existence of a twisted antipode $\widetilde{S}$ of square one is still a partial unimodularity condition on a Hopf algebra. It was crucial for the results of [CM] that this condition is actually fulfilled for the Hopf algebra associated to n-dimensional transverse geometry of foliations. Moreover, as we shall see now, it is also fulfilled by the most popular quantum groups.

First, recall that if $\mathcal{H}$ is *quasi-triangular* (also called *braided*), with universal R-matrix R, then (see e.g. [K])

$$S^2(x) = u\,x\,u^{-1} \ , x \in \mathcal{H}$$

with

$$u = \sum S(R^{(2)})\,R^{(1)} \ , \quad \varepsilon(u) = 1$$

and

$$\Delta\,u = (R_{21}\,R)^{-1}\,(u \otimes u)\,.$$

If in addition $u\,S(u) = S(u)\,u$, which is central, has a central square root θ, such that

$$\Delta(\theta) = (R_{21}\,R)^{-1}\,(\theta \otimes \theta) \ , \quad \varepsilon(\theta) = 1 \ , \quad S(\theta) = \theta\,,$$

then $\mathcal{H}$ is called a *ribbon* algebra. Any braided Hopf algebra $\mathcal{H}$ can be canonically embedded in a ribbon algebra (cf. [RT]):

$$\mathcal{H}(\theta) = \mathcal{H}\,[\theta]/(\theta^2 - u\,S(u))$$

If $\mathcal{H}$ is a ribbon algebra, then

$$\delta = \theta\,u^{-1}\,,$$

is a group-like element:

$$\Delta\,\delta = \delta \otimes \delta \ , \quad \varepsilon(\delta) = 1 \ , \quad S(\delta) = \delta^{-1}\,.$$

Defining

$$\widetilde{S} = \delta \cdot S\,,$$

one obtains a twisted antipode which satisfies the property $\widetilde{S}^2 = 1$. Indeed,

$$\begin{aligned} \widetilde{S}^2(x) &= \delta\,S(\delta\,S(x)) = \delta\,S^2(x)\,\delta^{-1} \\ &= \delta\,u\,x\,u^{-1}\,\delta^{-1} = \theta\,x\,\theta^{-1} = x\,, \end{aligned}$$

because θ is central.

By dualizing the above definitions one obtains the notion of a *cobraided*, resp. *coribbon* algebra. Among the most prominent examples of coribbon

algebras are the function algebras of the classical quantum groups $GL_q(N)$, $SL_q(N)$, $SO_q(N)$, $O_q(N)$ and $Sp_q(N)$.

For a coribbon algebra $\mathcal{H}$, the analogue of the above *ribbon group-like element* δ is the *ribbon character* $\delta \in \mathcal{H}^*$. The corresponding twisted antipode

$$\widetilde{S} = \delta * S\,,$$

satisfies again the condition $\widetilde{S}^2 = 1$.

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